ADVANCES IN MATHEMATICS 10, 387–394 (1973)

Zeta Functions for Abelian Characters of Simple Algebras

LAWRENCE CORWIN

Department of Mathematics, Yale University, New Haven, Connecticut

Let A be a central simple algebra of degree n^2 over an algebraic number field K, and denote its invertible elements by G. At a typical prime (finite or infinite) of K, A_v is the localization of A; it has a reduced norm $\nu_v = A_v \to K_v$. K_v has a valuation, $\| \ \|_v$, which we assume is the modulus function for Haar measure. We set $N_v(a_v) = \| \nu_v(a_v) \|_v$.

Let $\mathcal{A}$, $\mathcal{G}$ denote, respectively, the adele ring and idele group of A. Then $\mathcal{A}$ is isomorphic to its dual group $\mathcal{A}^\wedge$; we identify $\mathcal{A}$ and $\mathcal{A}^\wedge$ in such a way that Haar measure on $\mathcal{A}^\wedge$ is self-dual and $A^\perp = A$ (A is embedded in $\mathcal{A}$ via the diagonal map.) If $g = (g_v) \in \mathcal{G}$, we set $N(g) = (\prod_v N_v(g_v))$; then because of our choices of $\| \ \|_v$, $N_v(g) = 1$ if $g \in G$. (Again, we put G in $\mathcal{G}$ by using the diagonal map.) $\mathcal{G}$ is a unimodular group. See, e.g., [1, Section 4.2], for constructions and proofs.

Now let F be a "nice" (Schwartz class will do admirably) function on $\mathcal{A}$, and let ω be a bounded continuous function on $\mathcal{G}$. We define the zeta function $\zeta(F, \omega, s)$ by

$$\zeta(F, \omega, s) = \int_{\mathcal{G}} F(x)\, \omega(x)\, N(x)^s \, dx,$$

wherever the integral converges. (dx is the Haar measure on $\mathcal{G}$.)

The most important case of these ζ-functions arises when F and ω are themselves products of functions on the A_v and A_v^x, respectively; i.e., $F = \prod_v F_v$, $\omega = \prod_v \omega_v$. (In this case, F_v is the characteristic function of the integral elements of A_v for almost all v, and $\omega_v = 1$ on the unit elements of A_v for almost v.) Then

$$\zeta(F, \omega, s) = \prod_v \zeta_v(F_v, \omega, s),$$

where $\zeta_v(F_v, \omega_v, s) = \int_{A_v^x} F_v(x)\, \omega_v(x)\, N_v(x)^s \, dx$. ($dx$ is the Haar measure on A_v^x.)

It is not hard to show that the product converges wherever $\mathrm{Re}(s) > 1$

and defines an analytic function there. The question usually considered is whether $\zeta(F, \omega, s)$ can be continued analytically (or meromorphically) to all of $\mathbf{C}$. The usual method for answering this question is that of Iwasawa–Tate, spelled out in detail (for the case where A is a field) in Tate's thesis [5]. To apply this method, we must assume some sort of invariance of ω under G; we shall assume that $\omega(xg) = \omega(x)$ for all $x \in \mathscr{G}$, $g \in G$.

If A is not a division algebra, the method runs into some technical difficulties (as we shall see), because there are too many noninvertible elements in A. The devices for overcoming these difficulties have involved putting appropriate restrictions on ω and choosing functions F appropriately. In actual use, we are interested not in $\zeta(F, \omega, s)$, but rather in this function divided by finitely many "local factors" $\zeta_v(F_v, \omega_v, s)$. Thus, we have a certain amount of leeway in choosing F. This method has been exploited by Andrianov [1], who used it to show that $\zeta(F, \omega, s)$ has an analytic continuation and functional equation when ω is a spherical function on $\mathscr{G}$ (with respect to a certain compact subgroup) satisfying certain other conditions.

In this paper, we shall consider the case where ω is a character (i.e., multiplicative homomorphism into the complex numbers of norm 1). This case is not entirely contained in Andrianov's work and turns out to be fairly simple. Also, the method used gives a technical advantage since it allows us to say more about the existence of poles.

We begin with the basic tool of the Iwasawa–Tate method.

Lemma 1. *Let g, h be elements of $\mathscr{G}$, and let F be any sufficiently nice (Schwartz class, say) function on $\mathfrak{A}$. Then*

$$\sum_{a \in A} F(hag) = \sum_{a \in A} N(h^{-1}g^{-1})^n F^\wedge(g^{-1}ah^{-1}).$$

Proof. This is the Poisson summation formula (for the subgroup $A = A^\perp$) applied to $F_1(x) = F(hxg)$, since $F_1^\wedge(x) = N(h^{-1}g^{-1})^n F^\wedge(g^{-1}xh^{-1})$.

Next, we consider $\omega = \prod \omega_v$.

Lemma 2. Ker ν_v = *commutator subgroup of A^x; thus, $\omega_v(x_v)$ depends only on $\nu_v(x_v)$.*

Proof. See [6, pp. 323–324].

Let A^0 = set of singular elements of A, A_v^0 = set of singular elements of A_v.

LEMMA 3. *Let $\mathfrak{A}_0$ be any element of A^0, and let*

$$G_v^0 = \{x \in A_v^x : xa_0 = a_0\}.$$

If ω is trivial on G_v^0, then it is trivial.

Proof. Suppose $a_1 = g_1 a_0 g_2 (g_1, g_2$ nonsingular); the isotropy group of a_1 is $g_2^{-1} G_v^0 g_2$, and ω_v is trivial on $g_2^{-1} G_v^0 g_2 \Leftrightarrow \omega_v$ is trivial on G_v^0. Thus, we need to prove the lemma only for one matrix of each row rank (see [3, p. 46]). We let

$$a_0 = \begin{pmatrix} \epsilon_1 & & & \bigcirc \\ & \ddots & & \\ & & \epsilon_{r-1} & \\ \bigcirc & & & 0 \end{pmatrix},$$

where each $\epsilon_j = 0$ or 1; then if $b \in K^x$,

$$x_b = \begin{pmatrix} 1 & & & \bigcirc \\ & \ddots & & \\ & & 1 & \\ \bigcirc & & & b \end{pmatrix} \in G_v^0.$$

But $\nu_v(x_b) = b$; hence, ω_v is trivial on an element of determinant b; hence, ω_v is trivial.

Now we return to $\mathscr{G}$, let $\mathscr{G}_1 = \{x \in \mathscr{G} : N(x) = 1\}$. Pick an infinite valuation v_0, and let H be the set of elements $(h_v) \in \mathscr{G}$ with $h_v = 1$ for $v \notin v_0$, and $h_{v_0} = \alpha I$ $(\alpha > 0)$. Then H is a central subgroup of $\mathscr{G}$, and $\mathscr{G}_1 \times H = \mathscr{G}$. Suppose $x = (x_1, h)$, with $x_1 \in \mathscr{G}$, and $h \in H$ then $\omega(x) = \omega(x_1)\, \omega(h)$, and $N(x) = N(h)$. We shall assume that ω is trivial on H. This causes no loss of generality, since a character on H is a power of $N(h)$ and N is already present in the integral defining $\zeta(F, \omega, s)$.

LEMMA 4. *Let a_0 be an element of A_0, and let $\mathscr{G}_0 = \{x \in \mathscr{G}_1 : xa_0 = a_0\}$. If ω is trivial on $\mathscr{G}^0$, it is trivial on $\mathscr{G}_1$.*

Proof. Lemma 3 says that if $x = (x_v) \in \mathscr{G}_1$, then $\exists y = (y_v) \in \mathscr{G}^0$ with $\nu(y_v) = \nu(x_v)$, $\forall v$. Therefore, $\omega(x) = \omega(y) = 1$.

Now we are prepared to start on $\zeta(F, \omega, s)$. For $a > 0$, let h_a be the element of H with $N(h_a) = a$, and let

$$\zeta_a(F, \omega, s) = \int_{\mathscr{G}_1} F(h_a x)\, \omega(h_a x)\, N(h_a x)^s\, dx = \int_{\mathscr{G}_1} F(h_a x)\, \omega(x)\, a^s\, dx.$$

Then $\zeta(F, \omega, s) = \int_0^\infty \zeta_a(F, \omega, s)\, da/a$.

Let C be a measurable set of coset representatives for $\mathscr{G}_1/G$. Since $\mathscr{G}_1/G$ is compact (see, e.g., [2, p. 171-218]), C has a finite Haar measure.

LEMMA 5.

$$\zeta_a(F, \omega, s) + J_a(F, \omega, s) = \zeta_{a^{-1}}(F^\wedge, \omega^\wedge, n - s) + J'_{a^{-1}}(F, \omega, s),$$

where $J_a(F, \omega, s) = \sum_{\xi \in A_0} \int_C F(x_a x \xi)\, \omega(x)\, a^s\, dx$, $\omega^\wedge(x) = \omega(x)^{-1}$, *and* $J_a'(F, \omega, s) = \sum_{\xi \in A_0} \int_{C^{-1}} F^\wedge(\xi x x_a)\, \omega^\wedge(x)\, a^{n-s}\, dx$.

Proof.

$$\begin{aligned}
&\zeta_a(F, \omega, s) + J_a(F, \omega, s) \\
&= \int_{\mathscr{G}_1} F(x_a x)\, \omega(x_a x)\, a^s + \sum_{\xi \in A_0} \int_C F(x_a x \xi)\, \omega(x)\, a^s\, dx \\
&= \sum_{\xi \in A^x} \int_C F(x_a x \xi)\, \omega(x_a x \xi)\, a^s\, dx + \sum_{\xi \in A_0} \int_C F(x_a x \xi)\, \omega(x)\, a^s\, dx \\
&= \sum_{\xi \in A^x} \int_C F(x_a x \xi)\, \omega(x)\, a^s\, dx + \sum_{\xi \in A_0} \int_C F(x_a x \xi)\, \omega(x)\, a^s\, dx \\
&= \sum_{\xi \in A} \int_C F(x_a x \xi)\, \omega(x)\, a^s\, dx = \int_C \sum_{\xi \in A} F(x_a x \xi)\, \omega(x)\, a^s\, dx \\
&= \int_C \sum_{\xi \in A} F^\wedge(\xi x^{-1} x_a^{-1})\, N(x^{-1} x_a^{-1})^n\, \omega(x)\, a^s\, dx \\
&= \int_C \sum_{\xi \in A} F^\wedge(\xi x^{-1} x_a^{-1})\, \omega(x)\, a^{s-n}\, dx \\
&= \int_{C^{-1}} \sum_{\xi \in A} F^\wedge(\xi x x_a^{-1})\, \omega^\wedge(x)\, a^{s-n}\, dx \\
&= \sum_{\xi \in A} \int_{C^{-1}} F^\wedge(\xi x x_a^{-1})\, \omega^\wedge(x) (a^{-1})^{n-s}\, dx \\
&= \sum_{\xi \in A^x} \int_{C^{-1}} F^\wedge(\xi x x_a^{-1})\, \omega^\wedge(x) (a^{-1})^{n-s}\, dx \\
&\quad + \sum_{\xi \in A_0} \int_{C^{-1}} F^\wedge(\xi x x_a^{-1})\, \omega^\wedge(x) (a^{-1})^{n-s} \\
&= \int F^\wedge(\xi x x_a^{-1})\, \omega^\wedge(x) (a^{-1})^{n-s}\, dx + J'_{a^{-1}}(F, \omega, s) \\
&= \zeta_{a^{-1}}(F^\wedge, \omega^\wedge, n - s) + J'_{a^{-1}}(F, \omega, s).
\end{aligned}$$

Note that C^{-1} is a measurable set of coset representatives for $G/\mathscr{G}$.

LEMMA 6. *If ω is a character which is not identically* 1, *then*

$$J_a(F, \omega, s) = J_a'(F, \omega, s) = 0.$$

Proof. We consider J_a first. Let S be a set of representatives of the G-orbits in A_0. If $\xi_0 \in S$, let

$$P(\xi_0) = \sum_{\xi \in G\xi_0} \int_C F(x_a x \xi) \, \omega(x) \, dx.$$

It suffices to show that $P(\xi_0) = 0$ for all ξ_0.

Let G_0 be the subgroup of G fixing ξ_0. Then $F(x_a x \xi_0)$ and $\omega(x)$ are constant on the cosets xG_0, and thus we may regard them as defined on G_0. Next, let T be a set of coset representatives for G/G_0; then $T\xi_0 = G\xi_0$, and, hence,

$$P(\xi_0) = \sum_{g \in T} \int_C F(x_a x g \xi_0) \, \omega(x) \, dx = \int_{TC} F(x_a x g \xi_0) \, \omega(x) \, dx.$$

But TC is a measurable cross-section for $\mathscr{G}_1/G_0$; therefore, we have

$$P(\xi_0) = \int_{\mathscr{G}_1/G_0} F(x \xi_0 x_0) \, \omega(x) \, dx.$$

Let $\mathscr{G}_0$ be the isotropy group of ξ_0 in $\mathscr{G}_1$. Then $\mathscr{G}_0/G_0$ is compact, since $\mathscr{G}_0 \cap G_1 = G_0$, and, therefore, $\mathscr{G}_0/G_0$ is a closed set of the compact set $\mathscr{G}_1/G_1$. Moreover, F is constant on $\mathscr{G}_0$-cosets. Thus,

$$\begin{aligned} P(\xi_0) &= \int_{\mathscr{G}_1/\mathscr{G}_0} F(\bar{x} \xi_0 x_0) \left[\int_{\mathscr{G}_0/G_0} \omega(\bar{x} y) \, dy \right] d\bar{x} \\ &= \int_{\mathscr{G}_1/\mathscr{G}_0} F(\bar{x} \xi_0 x_a) \, \omega(\bar{x}) \, d\bar{x} \int_{\mathscr{G}_0/G_0} \omega(y) \, dy. \end{aligned}$$

(All the groups are unimodular; an argument showing this for $\mathscr{G}_0$ will be given later.)

The character $\omega(y)$ is nontrivial on $\mathscr{G}_0$ by Lemma 4. But if $\omega(y_0) \neq 1$, then

$$\int_{\mathscr{G}_0/G_0} \omega(y) \, dy = \int_{\mathscr{G}_0/G_0} \omega(y y_0) \, dy = \omega(y_0) \int_{\mathscr{G}/\mathscr{G}_0} \omega(y) \, dy,$$

and so

$$\int_{\mathscr{G}_0/G_0} \omega(y) \, dy = 0.$$

Hence, $P(\xi_0) = 0$, as desired.

The argument for J' is similar; we need to use the analogue of Lemma 4 for right actions of G.

THEOREM. *If ω is a character which is not identically 1, then $\zeta(F, \omega, s)$ extends to an analytic function defined on all of $\mathbf{C}$. Moreover, it satisfies the functional equation*

$$\zeta(F, \omega, s) = \zeta(F^\wedge, \omega^\wedge, n - s),$$

where

$$n = [A: K].$$

Proof (as in Tate [5]). In view of the two previous lemmas, $\zeta_a(F, \omega, s) = \zeta_{a^{-1}}(F^\wedge, \omega^\wedge, n - s)$.
Therefore,

$$\begin{aligned}\zeta(F, \omega, s) &= \int_0^1 \zeta_a(F, \omega, s) \frac{da}{a} + \int_1^\infty \zeta_a(F, \omega, s) \frac{da}{a} \\ &= \int_0^1 \zeta_{a^{-1}}(F^\wedge, \omega^\wedge, n - s) \frac{da}{a} + \int_1^\infty \zeta_a(F, \omega, s) \frac{da}{a} \\ &= \int_1^\infty [\zeta_a(F^\wedge, \omega^\wedge: n - s) + \zeta_a(F, \omega, s)] \frac{da}{a}.\end{aligned}$$

Since $\int_0^\infty \zeta_a(F, \omega, s)\, da/a$ converges for $\mathrm{Re}(s) > 1$, the integral from 1 to ∞ converges for $\mathrm{Re}(s) > 1$ and therefore (by dominated convergence) for all s. Similarly, $\int_1^\infty \zeta_a(F^\wedge, \omega^\wedge, n - s)\, da/a$ exists for all s. We, thus, have $\zeta(F, \omega, s)$ written as an everywhere convergent integral. This expression is obviously analytic in s and symmetric under $F \mapsto F^\wedge$, $\omega \mapsto \omega^\wedge$, $s \mapsto n - s$. This proves the theorem.

It may be worthwhile noting what distinguishes the previous approach from that of Andrianov [1]. First of all, his ω's are not necessarily constant on G-cosets of $\mathscr{G}$, but are integrals of functions which are. This introduces a slight complication, but does not really change matters. His main problem, too, is getting rid of $J_a(F, \omega, s)$. His solution is to choose F so that if $\xi \in A_0$, then $F(x\xi) = F^\wedge(\xi x) = 0$, but so that $\zeta(F, \omega, s)$ is not identically 0. The result is that J and J' disappear automatically. The price he pays is that some of his "local" ζ-functions $\zeta_v(F, \omega, s)$ have zeroes in the interior of the critical strip $0 \leqslant \mathrm{Re}(s) \leqslant n$. In practice, we are usually interested not in $\zeta(F, \omega, s)$, but in it divided by finitely many of the ζ_v; it is difficult to decide, using Andrianov's method, where the poles of this function are. The foregoing method places essentially no restrictions on F. For the computations involved in the local ζ-functions, see [7, Chapter X, Section 3; 5, Chapter 2].

We still have two loose ends to clean up. One is that of analytic continuation when $\omega = 1$. The previous method does not work, because the J_a's do not disappear. However, we may use a trick to take care of this case. At almost every finite prime v, A_v is unramified over $\underset{\sim}{Q}$ and F_v is the characteristic function of the integers of A_v. For these v, $A_v = M_n(P_v)$, and

$$\zeta_v(F_v, \omega_v, s) = \prod_{j=0}^{n-1} (1 - q_v^{j-s})^{-1},$$

where q_v is the order of the residue class field of K_v. (See [7, p. 197]). The field K also has a local zeta-funtion with respect to f, the characteristic function of its integers, and χ_v, the trivial character; it is $\zeta_v(f_v, \chi_v, s) = (1 - q_v^{-s})^{-1}$. Therefore,

$$\zeta_v(F_v, \omega_v, s) = \prod_{j=0}^{n-1} \zeta_v(f_v, \chi_v, s - j).$$

Hence,

$$\zeta(F_v, \omega_v, s) = R(s) \prod_{j=0}^{n-1} \zeta(f, \chi, s - j),$$

where $R(s)$ is a finite product of meromorphic functions (from the other valuations). But the functional equation for $\zeta(f, \chi, s - j)$ was done by Tate [5]. In our notation, $K_0 = \{0\}$, and so $J_a(f, \chi, s) = \int_C f(0)\, \lambda(x)\, a^s\, dx$. Since $\chi \equiv 1$, $J_a(f, \chi, s) = cf(0)\, a^s$, where c is the volume of C. Similarly, $J_a'(f, \chi, s) = cf\hat{}(0)\, a^{1-s}$. We can now carry these terms though the proof of the theorem.

Finally, we need to show that $\mathscr{G}_0$ is unimodular. Suppose that $\xi_0 = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where I is a $(n - k) \times (n - k)$ matrix. Then at each prime, an element of $\mathscr{G}_0$ looks like $\left(\begin{smallmatrix} I & 0 \\ A_v & B_v \end{smallmatrix}\right)$, where B_v is $k \times k$ invertible matrix. Therefore, $\mathscr{G}_0$ is the semidirect product of $\mathscr{N}$ by $\mathscr{H}$, where $\mathscr{N}$ consists of ideles which (at each prime v) look like $\left(\begin{smallmatrix} I & 0 \\ A_v & I \end{smallmatrix}\right)$, and $\mathscr{H}$ consists of ideles h looking at each prime v like $\left(\begin{smallmatrix} I & 0 \\ 0 & B_v \end{smallmatrix}\right)$, and which satisfy $N(h) = 1$. Both $\mathscr{N}$ and $\mathscr{H}$ are unimodular; we can now apply [3, 15.29, p. 210], to compute the modular function of $\mathscr{G}_0$, and we find that it is 1.

References

1. A. Andrianov, Zeta-functions of simple algebras with non-abelian characters, *Russ. Math. Surveys* **23** (1968), 1–65.

2. R. GODEMENT, Les fonctions ζ des algèbres simples I, *Séminaire Bourbaki* **11**, Exposé No. 171 (1959).
3. E. HEWITT AND K. A. ROSS, "Abstract Harmonic Analysis I," Springer-Verlag, New York, 1963.
4. N. JACOBSON, "Lectures in Abstract Algebra," vol. 2, Van Nostrand, Princeton, NJ, 1953
5. J. T. TATE, Fourier analysis in number fields and Hecke's ζ-functions, Thesis, Princeton Univ., 1950; reprinted in "Algebraic Number Theory" (J. Cassels and A. Fröhlich, Eds.), pp. 305–346, Thompson Book Co., Washington, D.C., 1967.
6. S. Wang, On the commutator group of a simple algebra, *Amer. J. Math.* **72** (1950), 323–334.
7. A. WEIL, "Basic Number Theory," Springer-Verlag, New York, 1967.